Department of Mathematics & Philosophy of Engineering

Faculty of Engineering Technology

The Open University of Sri Lanka –Nawala, Nuggegoda

Course: MPZ 3132 - Engineering Mathematics IB

Model Answer No.01

Academic Year 2012/2013

These are the common solutions for the assignment No. 1. You can find your solutions after substituting your values of a, b and c.

1.

Drichelet's condition for a Fourier series

If the function $f(x)$ for the interval $(-\pi, \pi)$

a. is single valued

b. is bounded

c. has at most finite number of maxima and minima

d. has only a finite number of discontinues

e. is $f(x+2\pi) = f(x)$ for values of x outside $(-\pi, \pi)$, then

$$S_p = \frac{a_0}{2} + \sum_{n=1}^{p} (a_n cosnx + b_n sinnx)$$

converges to $f(x)$ as $p \to \infty$ at the values of x for which $f(x)$ is continuous and $\frac{1}{2}\{f(x+0) + f(x-0)]$

1.

$f(x) = a + 2bx + 3cx^2 \ , -\pi << \pi$ *The Fourier series of* $f(x)$ *with period* 2π *is*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n cos\left(\frac{n\pi x}{l}\right) + b_n sin\left(\frac{n\pi x}{l}\right)\right)$$

$$a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)dx \, , a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)cos(nx)dx \, , \ b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)sin(nx)dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(a + 2bx + 3cx^2)dx = \frac{1}{\pi}[ax + bx^2 + cx^3]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}[a\pi + b\pi^2 + c\pi^3] - \frac{1}{\pi}[-a\pi + b\pi^2 - c\pi^3] = 2[a + c\pi^2]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx = \frac{1}{\pi}\int_{-\pi}^{\pi}(a + 2bx + 3cx^2)cosnxdx$$

$$= \frac{1}{\pi}\left[(a + 2bx + 3cx^2)\frac{sinnx}{n} + (2b + 6cx)\frac{cosnx}{n^2} - 6c\frac{sinnx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(2b + 6c\pi)\frac{(-1)^n}{n^2}\right] - \frac{1}{\pi}\left[(2b - 6c\pi)\frac{(-1)^n}{n^2}\right] = \frac{12c(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx = \frac{1}{\pi}\int_{-\pi}^{\pi}(a + 2bx + 3cx^2)sinnxdx$$

$$= \frac{1}{\pi}\left[(a + 2bx + 3cx^2)\left(-\frac{cosnx}{n}\right) + (2b + 6cx)\frac{sinnx}{n^2} + 6c\frac{cosnx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(a + 2b\pi + 3c\pi^2)\left(-\frac{(-1)^n}{n}\right) + 6c\frac{(-1)^n}{n^3}\right]$$

$$-\frac{1}{\pi}\left[(a - 2b\pi + 3c\pi^2)\left(-\frac{(-1)^n}{n}\right) + 6c\frac{(-1)^n}{n^3}\right] = \frac{-4b(-1)^n}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx)$$

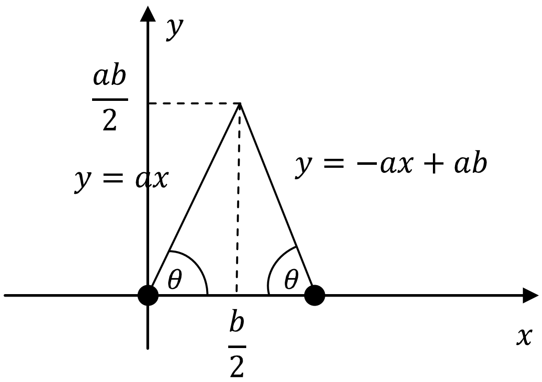
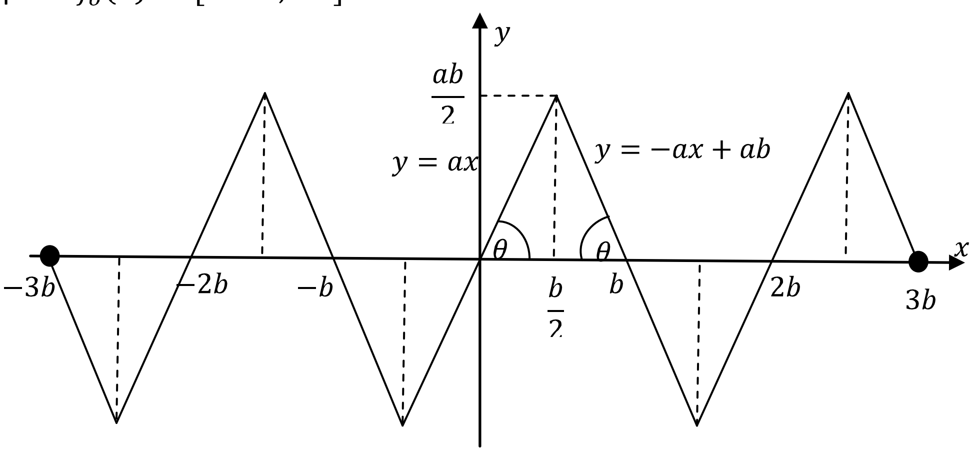
$$a + 2bx + 3cx^2 = a + c\pi^2 + \sum_{n=1}^{\infty}\left(\frac{12c(-1)^n}{n^2}cosnx + \frac{-4b(-1)^n}{n}sinnx\right)$$

$$= a + c\pi^2 + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n}\left(\frac{3c}{n}cosnx - bsinnx\right)$$

$$\lim_{x\to\pi^+} f(x) = a - 2b\pi + 3c\pi^2 \ and \ \lim_{x\to\pi^-} f(x) = a + 2b\pi + 3c\pi^2$$

$$\frac{1}{2}\left[\lim_{x\to\pi^+} f(x) + \lim_{x\to\pi^-} f(x)\right] = a + 3c\pi^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

	$$\frac{1}{2}\left[\lim_{x\to\pi^+} f(x)+\lim_{x\to\pi^-} f(x)\right]=a+c\pi^2+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n}\left(\frac{3c}{n}cosn\pi-bsinn\pi\right)$$ $$=a+c\pi^2+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}3c(-1)^n=a+c\pi^2+12c\sum_{n=1}^{\infty}\frac{1}{n^2}\ldots\ldots\ldots\ldots(2)$$ Therefore by (1) and (2) $we\ get$ $$a+c\pi^2+12c\sum_{n=1}^{\infty}\frac{1}{n^2}=a+3c\pi^2\quad\therefore\ \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$	

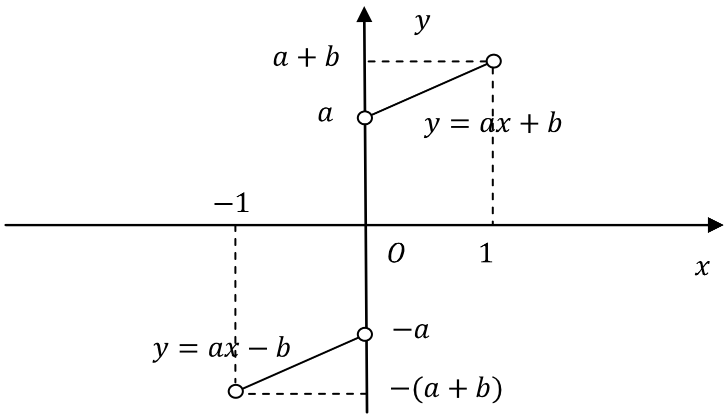
2.		
2.1	The graph of $f(x)$ is 	
	The extended odd function is $$f(x)=\begin{cases}a(b-x) & when\ \frac{b}{2}\le x\le b\\ ax & when\ -\frac{b}{2}\le x\le\frac{b}{2}\\ a(b+x) & when\ -b\le x\le-\frac{b}{2}\end{cases}$$ $$f_o(x)=f_o(x+2bk)\ \ where\ k\in Z$$	
2.2	The graph of $f_o(x)$ in $[-3b, 3b]$ is 	

2.3	Since $f_o(x)$ is odd function $a_0 = 0 \; and \; a_n = 0$ $$f_o(x) = \sum_{n=1}^{\infty} b_n sin\left(\frac{n\pi x}{l}\right)$$ $$b_n = \frac{2}{b}\int_0^b f_o(x) sin\left(\frac{n\pi x}{b}\right)dx = \frac{2}{b}\int_0^{\frac{b}{2}} axsin\left(\frac{n\pi x}{b}\right)dx + \frac{2}{b}\int_{\frac{b}{2}}^{b} a(b-x)sin\left(\frac{n\pi x}{b}\right)dx$$ $$= \frac{2}{b}\left[ax\left(\frac{-cos\frac{n\pi x}{b}}{\frac{n\pi}{b}}\right) + a\left(\frac{sin\frac{n\pi x}{b}}{\left(\frac{n\pi}{b}\right)^2}\right)\right]_0^{\frac{b}{2}}$$ $$+\frac{2}{b}\left[a(b-x)\left(\frac{-cos\frac{n\pi x}{b}}{\frac{n\pi}{b}}\right) - a\left(\frac{sin\frac{n\pi x}{b}}{\left(\frac{n\pi}{b}\right)^2}\right)\right]_{\frac{b}{2}}^{b}$$ $$= \frac{2}{b}\left[-a\frac{b}{2}\frac{b}{n\pi}cos\frac{n\pi}{2} + a\frac{b^2}{n^2\pi^2}sin\frac{n\pi}{2} + a\frac{b}{2}\frac{b}{n\pi}cos\frac{n\pi}{2} + a\frac{b^2}{n^2\pi^2}sin\frac{n\pi}{2}\right]$$ $$= \frac{4ab}{n^2\pi^2}sin\frac{n\pi}{2}$$ Suppose n is odd then $n = 2m+1 \; where \; m \in Z$ $$sin\frac{n\pi}{2} = sin\frac{(2m+1)\pi}{2} = (-1)^m$$ Suppose n is even then $n = 2m \; where \; m \in Z$ $$sin\frac{n\pi}{2} = \sin(m\pi) = 0$$ $$f_o(x) = \frac{4ab}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^m}{(2m+1)^2}sin\left(\frac{(2m+1)\pi x}{b}\right)$$	
	$$When \;\; \frac{b}{2} \;\; f_o(x) = \frac{ab}{2}$$ $$\frac{ab}{2} = \frac{4ab}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^m}{(2m+1)^2}sin\left(\frac{(2m+1)}{1}\frac{\pi}{2}\right)$$ $$\frac{ab}{2} = \frac{4ab}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^m}{(2m+1)^2}(-1)^m \quad \therefore \; \frac{1}{2} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2m+1)^2}$$ $$\sum_{n=1}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}$$	

3.

3.1 The graph of a odd function is skew symmetric about y – axis

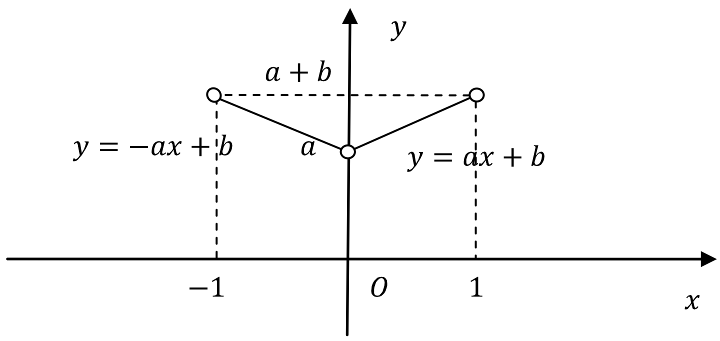
3.1.1



$$f_o(x) = \begin{cases} ax + b & when \quad 0 < x < 1 \\ ax - b & when \ -1 < x < 0 \end{cases}$$

Other values of x $f_o(x) = f_o(x + 2k)$ $where$ $k \in Z$

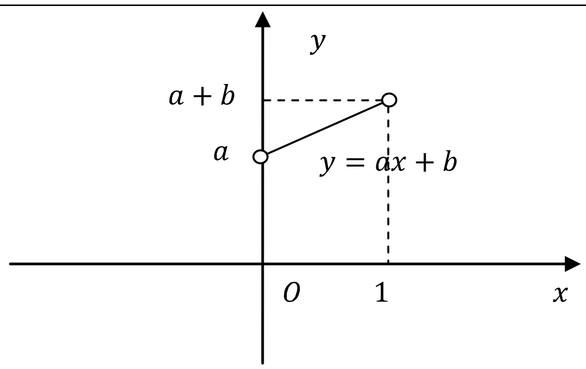
3.1.2 The graph of a odd function is symmetric about y – axis



$$f_e(x) = \begin{cases} ax + b & when \quad 0 < x < 1 \\ -ax + b & when \ -1 < x < 0 \end{cases}$$

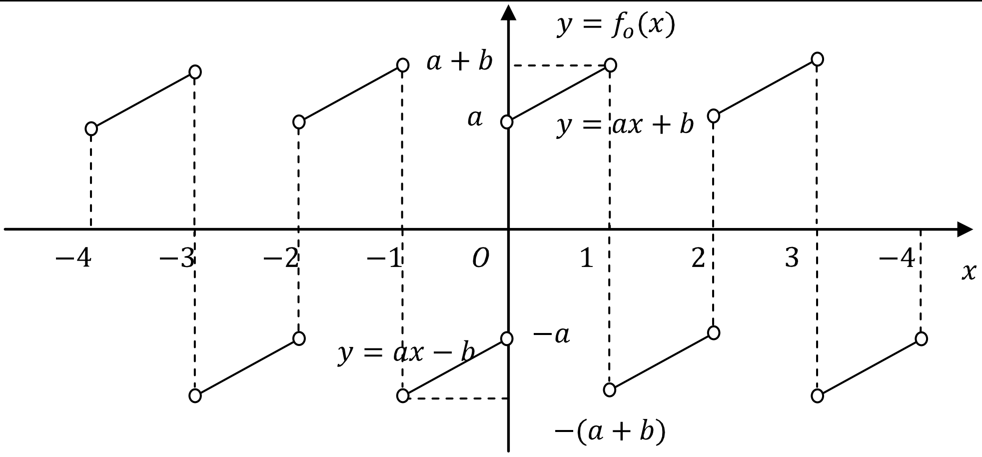
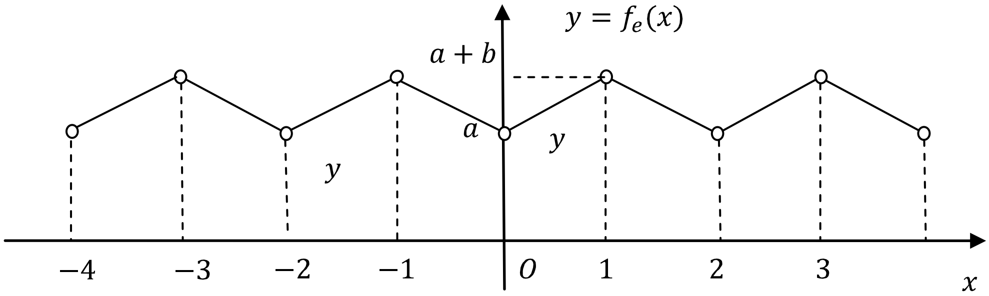
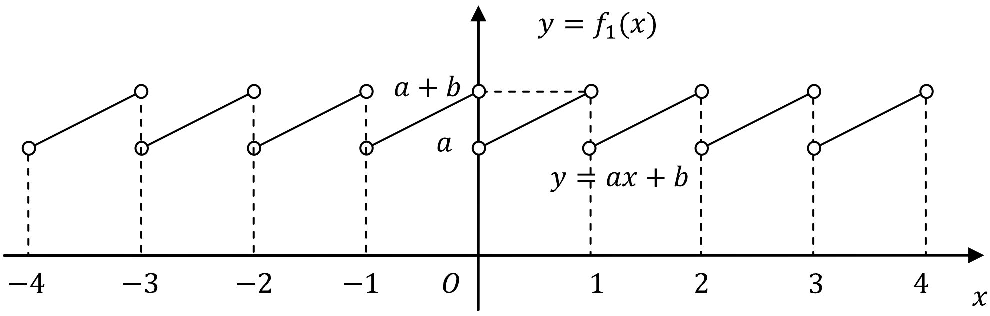
Other values of x $f_o(x) = f_o(x + 2k)$ $where$ $k \in Z$

3.1.3



$$f_1(x) = ax + b \quad when \quad 0 < x < 1$$

Other values of x $f_1(x) = f_1(x + k)$ $where$ $k \in Z$

<table>
<tr><td>3.2</td><td></td><td></td></tr>
<tr><td>3.2.1</td><td>
</td><td></td></tr>
<tr><td>3.2.2</td><td>
</td><td></td></tr>
<tr><td>3.2.3</td><td>
</td><td></td></tr>
<tr><td>3.3
3.3.1</td><td>

Since $f_o(x)$ is odd function $a_0 = 0$ and $a_n = 0$

$$b_n = \frac{2}{l}\int_0^l f(x)sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{1}\int_0^1 (ax+b)sin\left(\frac{n\pi x}{1}\right)dx$$

$$= 2\left[(ax+b)\left(\frac{-cos(n\pi x)}{n\pi}\right) - a\left(\frac{-sin(n\pi x)}{(n\pi)^2}\right)\right]_0^1$$

$$= 2\left[-(a+b)\left(\frac{(-1)^n}{n\pi}\right) + b\left(\frac{1}{n\pi}\right)\right]_0^1 = \frac{2}{n\pi}[b-(a+b)(-1)^n]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n cos\left(\frac{n\pi x}{l}\right) + b_n sin\left(\frac{n\pi x}{l}\right)\right)$$

$$ax + b = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}[b-(a+b)(-1)^n]sin(n\pi x)\right)$$

</td><td></td></tr>
</table>

3.3.2	Since $f_e(x)$ is even function $b_n = 0$ $$a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{1}\int_0^1 (ax+b)dx = 2\left[\frac{ax^2}{2} + bx\right]_0^1 = a + 2b$$ $$a_n = \frac{2}{l}\int_0^l f(x)cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{1}\int_0^1 (ax+b)cos\left(\frac{n\pi x}{1}\right)dx$$ $$= 2\left[(ax+b)\left(\frac{sin(n\pi x)}{n\pi}\right) - a\left(\frac{cos(n\pi x)}{(n\pi)^2}\right)\right]_0^1 = 2\left[a\left(\frac{(-1)^n}{n^2\pi^2}\right) + a\left(\frac{1}{n^2\pi^2}\right)\right]$$ $$= \frac{2a}{n^2\pi^2}[(-1)^n + 1]$$ $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n cos\left(\frac{n\pi x}{l}\right) + b_n sin\left(\frac{n\pi x}{l}\right)\right)$$ $$ax + b = \frac{a}{2} + b + \frac{2a}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}[(-1)^n + 1]cos(n\pi x)\right)$$	
3.3.3	$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n cos\left(\frac{n\pi x}{l}\right) + b_n sin\left(\frac{n\pi x}{l}\right)\right)$$ $$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx\, , a_n = \frac{1}{l}\int_c^{c+2l} f(x)cos\left(\frac{n\pi x}{l}\right)dx\, ,\ b_n$$ $$= \frac{1}{l}\int_c^{c+2l} f(x)sin\left(\frac{n\pi x}{l}\right)dx$$ $$f_1(x) = ax + b \quad when \qquad 0 < x < 1$$ Other values of $x \quad f_1(x) = f_1(x+k) \;\; where \;\; k \in Z$ $$a_0 = \frac{1}{\frac{1}{2}}\int_0^1 (ax+b)dx = 2\left[\frac{ax^2}{2} + bx\right]_0^1 = a + 2b$$ $$a_n = \frac{1}{\frac{1}{2}}\int_0^1 (ax+b)cos\left(\frac{n\pi x}{\frac{1}{2}}\right)dx$$ $$= 2\left[(ax+b)\left(\frac{sin(2n\pi x)}{2n\pi}\right) - a\left(\frac{cos(2n\pi x)}{(2n\pi)^2}\right)\right]_0^1 = 0$$	

	$$b_n = \frac{1}{\frac{1}{2}} \int_0^1 (ax+b) sin\left(\frac{n\pi x}{\frac{1}{2}}\right) dx$$ $$= 2\left[(ax+b)\left(-\frac{cos(2n\pi x)}{2n\pi}\right) + a\left(\frac{sin(2n\pi x)}{(2n\pi)^2}\right)\right]_0^1$$ $$2\left[-(a+b)\frac{1}{2n\pi} + \frac{b}{2n\pi}\right] = -\frac{a}{n\pi}$$ $$ax+b = \frac{a}{2} + b + \sum_{n=1}^{\infty}\left(-\frac{a}{n\pi}\right) sin\left(\frac{n\pi x}{\frac{1}{2}}\right) = \frac{a}{2} + b - \frac{a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\,(2n\pi x)$$	

4.	If $f^r(k)$ exists for all $r \in N$ $$f(x) = \sum_{r=0}^{\infty} \frac{f^r(k)}{r!}(x-k)^r$$	
4.1	$$f(x) = (a^2x + b^2)^{-1}$$ $$f^1(x) = (-1)(a^2x+b^2)^{-2}a^2$$ $$f^2(x) = (-1)(-2)(a^2x+b^2)^{-3}(a^2)^2$$ $$f^3(x) = (-1)(-2)(-3)(a^2x+b^2)^{-4}(a^2)^3$$ Similarly in the above manner $$f^n(x) = (-1)(-2)(-3)\ldots\ldots(-n)(a^2x+b^2)^{-(n+1)}(a^2)^n$$ $$f^n(x) = \frac{(-1)^n n!\, a^{2n}}{(a^2x+b^2)^{n+1}}$$	
4.2	$$\therefore \quad f^n(0) = \frac{(-1)^n n!\, a^{2n}}{(b^2)^{n+1}}$$ $$\frac{1}{a^2x+b^2} = \sum_{n=0}^{\infty} \frac{(-1)^n n!\, a^{2n}}{(b^2)^{n+1} n!} x^n$$ $$\frac{1}{a^2x+b^2} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{b^{2n+2}} x^n$$	
4.3	x^2 is substituted instead of x $$\frac{1}{a^2x^2+b^2} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{b^{2n+2}} x^{2n}$$	

4.4	By integrating both sides $$\int \frac{1}{a^2x^2+b^2}\,\mathrm{dx} = \int \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{b^{2n+2}} x^{2n} dx + Const.$$ When $x = 0$ $Const. = 0$ $$\int \frac{1}{a^2x^2+b^2}\,\mathrm{dx} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{b^{2n+2}} \int x^{2n}\, dx$$ $$\frac{1}{a^2}\int \frac{1}{x^2+\frac{b^2}{a^2}}\,\mathrm{dx} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{b^{2n+2}} \frac{x^{2n+1}}{(2n+1)}$$ $$\frac{1}{a^2}\frac{1}{\frac{b}{a}} tan^{-1}\left(\frac{x}{\frac{b}{a}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{b^{2n+2}} \frac{x^{2n+1}}{(2n+1)}$$ $$\frac{1}{ab} tan^{-1}\left(\frac{ax}{b}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{b^{2n+2}} \frac{x^{2n+1}}{(2n+1)}$$	
4.5	$$\frac{1}{a^2x^2+b^2} = \frac{1}{\mathrm{b}^2} + \sum_{n=1}^{\infty} \frac{(-1)^n a^{2n}}{b^{2n+2}} x^{2n}$$ Differentiating both sides $$\frac{d}{dx}(a^2x^2+b^2)^{-1} = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=1}^{\infty} \frac{(-1)^n a^{2n}}{b^{2n+2}} x^{2n}$$ $$-(a^2x^2+b^2)^{-2}a^2 2x = \sum_{n=1}^{\infty} \frac{(-1)^n a^{2n}}{b^{2n+2}} \frac{\mathrm{d}}{\mathrm{d}x} x^{2n}$$ $$\frac{2a^2\mathrm{x}}{(a^2x^2+b^2)^2} = \sum_{n=1}^{\infty} \frac{2n(-1)^{n-1} a^{2n}}{b^{2n+2}} x^{2n-1}$$	

5.	If $f^r(k)$ exists for all $r \in \{0\ ,1\ .2\ .3\ \dots\dots\dots n\}$ $$f(x)=\sum_{r=0}^{n}\frac{f^r(k)}{r!}(x-k)^r$$	
5.1	$$f(0)=a\ \ and\ \ \frac{df(x)}{dx}=bxf(x)+a$$ $$f^1(x)=bxf(x)+a \therefore\ f^1(0)=a$$ $$f^2(x)=bxf^1(x)+bf(x)\quad \therefore\ f^2(0)=bf(0)=\mathrm{ab}$$ $$f^3(x)=bxf^2(x)+bf^1(x)+bf^1(x)=bxf^2(x)+2bf^1(x)$$ $$\therefore\ f^3(0)=2bf^1(0)=2\mathrm{ab}$$ $$f^4(x)=bxf^3(x)+bf^2(x)+2bf^2(x)=bxf^3(x)+3bf^2(x)$$ $$\therefore\ f^4(0)=2bf^2(0)=3\mathrm{ab}^2$$ $$f(x)=f(0)+\frac{f^1(0)}{1!}x+\frac{f^2(0)}{2!}x^2+\frac{f^3(0)}{3!}x^3+\frac{f^4(0)}{4!}x^4$$ $$f(x)=a+\frac{a}{1!}x+\frac{ab}{2!}x^2+\frac{2ab}{3!}x^3+\frac{3ab^2}{4!}x^4$$ $$f(x)=a+ax+\frac{ab}{2}x^2+\frac{ab}{3}x^3+\frac{ab^2}{8}x^4$$	
5.2	$$f(x)=\sum_{r=0}^{n}\frac{f^r(k)}{r!}(x-k)^r$$ $$If\ f(x)=sinx\ ,\ \ f^1(x)=cosx\ \ ,f^2(x)=-sinx\ ,$$ $$f^3(x)=-cosx\ ,f^4(x)=sinx$$ $$f^5(x)=cosx\ ,\ \ f^6(x)=-sinx,\ \ f^7(x)=-cosx$$ $$If\ f(k)=sink\ ,\ \ f^1(k)=cosk\ \ ,f^2(k)=-sink\ ,f^3(k)=-cosk\ ,f^4(k)=sink$$ $$f^5(k)=cosk\ ,\ \ f^6(k)=-sink,\ \ f^7(k)=-cosk$$ $$\sin(k+x)=sink+\frac{cosk}{1!}x-\frac{sink}{2!}x^2-\frac{cosk}{3!}x^3+\frac{sink}{4!}x^4+\frac{cosk}{5!}x^5-\frac{sink}{6!}x^6-\frac{cosk}{7!}x^7$$	

	$6a^0 = 60^0 + a^0 = \frac{\pi}{3} + \frac{\pi a}{180}$ $when \ \ k = \frac{\pi}{3} \ and \ x = \frac{\pi a}{180}$ $\sin 6a^0 = sin\frac{\pi}{3} + \frac{cos\frac{\pi}{3}}{1!}\left(\frac{\pi a}{180}\right) - \frac{sin\frac{\pi}{3}}{2!}\left(\frac{\pi a}{180}\right)^2 - \frac{cos\frac{\pi}{3}}{3!}\left(\frac{\pi a}{180}\right)^3$ $+ \frac{sin\frac{\pi}{3}}{4!}\left(\frac{\pi a}{180}\right)^4 + \frac{cos\frac{\pi}{3}}{5!}\left(\frac{\pi a}{180}\right)^5 - \frac{sin\frac{\pi}{3}}{6!}\left(\frac{\pi a}{180}\right)^6 - \frac{cos\frac{\pi}{3}}{7!}\left(\frac{\pi a}{180}\right)^7$ $\sin 6a^0 = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{\pi a}{180}\right) - \frac{\sqrt{3}}{4}\left(\frac{\pi a}{180}\right)^2 - \frac{1}{12}\left(\frac{\pi a}{180}\right)^3$ $+ \frac{\sqrt{3}}{48}\left(\frac{\pi a}{180}\right)^4 + \frac{1}{240}\left(\frac{\pi a}{180}\right)^5 - \frac{\sqrt{3}}{6!}\left(\frac{\pi a}{180}\right)^6 - \frac{1}{7!}\left(\frac{\pi a}{180}\right)^7$ if $a = 1$ $sin61^0 = 0.866025404 + 0.008726647 - 0.000131903 - 0.000000443$ $+ 0.000000003 + 0.000000000 = 0.874619708$ if $a = 2$ $sin62^0 = 0.866025404 + 0.017453293 - 0.000527613 - 0.000003544$ $+ 0.000000054 + 0.000000000 = 0.882947594$	